arXiv:1008.1209v1 [math.CO] 6 Aug 2010

Distance-regular graphs with large a_1 or c_2

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August 9, 2010

Abstract

In this paper, we study distance-regular graphs Γ that have a pair of distinct vertices, say x and y, such that the number of common neighbors of x and y is about half the valency of Γ. We show that if the diameter is at least three, then such a graph, besides a finite number of exceptions, is a Taylor graph, bipartite with diameter three or a line graph.

Key Words: distance-regular graphs; Taylor graphs
2000 Mathematics Subject Classification: 05E30

1 Introduction

In this paper, we study distance-regular graphs Γ that have a pair of distinct vertices, say x and y, such that the number of common neighbors of x and y is about half the valency of Γ.

To be more precise, let Γ be a distance-regular graph with valency k and diameter D. If x and y are adjacent vertices (respectively vertices at distance two), then $a_1 := a_1(x, y)$ (respectively $c_2 := c_2(x, y)$) denotes the number of common neighbors of x and y. It is known that $a_1(x, y)$ (respectively $c_2(x, y)$) does not depend on the specific pair of vertices x and y at distance one (respectively two).

Brouwer and Koolen [8] showed that if $c_2 > \frac{1}{2}k$, then $D \leq 3$, and $D = 3$ implies that Γ is either bipartite or a Taylor graph. In Proposition 5, we slightly extend this result.

*This work was partially supported by the Priority Research Centers Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant number 2009-0094069). JHK was also partially supported by the Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (Grant number 2009-0089826).

In Theorem 11, we classify the distance-regular graphs with $a_1 \geq \frac{1}{2}k - 1$ and diameter at least three. Besides the distance-regular line graphs (classified by Mohar and Shawe-Taylor [13], cf. [6, Theorem 4.2.16]) and the Taylor graphs, one only finds the Johnson graph $J(7,3)$ and the halved 7-cube. This is in some sense a generalization of the classification of the claw-free distance-regular graphs of Blokhuis and Brouwer [4], as the claw-freeness condition implies $a_1 \geq \frac{1}{2}k - 1$. But they also classified the claw-free connected non-complete strongly regular graphs (i.e. distance-regular graphs with diameter two). The classification of strongly regular graphs with $a_1 \geq \frac{1}{2}k - 1$ seems to be hopeless, as there are infinitely many strongly regular graphs that satisfy $a_1 \geq \frac{1}{2}k - 1$, but are not line graphs, for example all the Paley graphs. A similar situation holds for the Taylor graphs. Note that the distance two graph of a non-bipartite Taylor graph is also a Taylor graph, and hence at least one of them has $a_1 \geq \frac{k-1}{2}$ and the other one has $c_2 \geq \frac{k-1}{2}$. If $a_1 = \frac{k-1}{2}$, then it is locally a conference graph, and for any conference graph Δ, we have a Taylor graph which is locally Δ (see, for example [6, Theorem 1.5.3]). Also there are infinitely many Taylor graphs with $a_1 \geq \frac{1}{2}k$, and hence infinitely many Taylor graphs with $c_2 \geq \frac{1}{2}k$ (see, for example [6, Theorem 1.5.3] and [9, Lemma 10.12.1]).

In the last section of this paper, we will discuss distance-regular graphs with $k_2 < 2k$, where k_2 is the number of vertices at distance two from a fixed vertex. In particular, in Theorem 12, we show that for fixed $\varepsilon > 0$, there are only finitely many distance-regular graphs with diameter at least three and $k_2 \leq (2-\varepsilon)k$, besides the polygons and the Taylor graphs. In Theorem 13, we classify the distance-regular graphs with $k_2 \leq \frac{3}{2}k$.

2 Definitions and preliminaries

All the graphs considered in this paper are finite, undirected and simple (for unexplained terminology and more details, see [6]). Suppose that Γ is a connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, where $E(\Gamma)$ consists of unordered pairs of two adjacent vertices. The *distance* $d(x,y)$ between any two vertices x and y of Γ is the length of a shortest path connecting x and y in Γ. We denote v as the number of vertices of Γ and define the *diameter* D of Γ as the maximum distance in Γ. For a vertex $x \in V(\Gamma)$, define $\Gamma_i(x)$ to be the set of vertices which are at distance precisely i from x $(0 \leq i \leq D)$. In addition, define $\Gamma_{-1}(x) = \Gamma_{D+1}(x) := \emptyset$. We write $\Gamma(x)$ instead of $\Gamma_1(x)$ and define the *local graph* $\Delta(x)$ at a vertex $x \in V(\Gamma)$ as the subgraph induced on $\Gamma(x)$. Let Δ be a graph. If the local graph $\Delta(x)$ is isomorphic to Δ for any vertex $x \in \Gamma(x)$, then we say Γ is locally Δ.

A connected graph Γ with diameter D is called *distance-regular* if there are integers b_i, c_i $(0 \leq i \leq D)$ such that for any two vertices $x, y \in V(\Gamma)$ with $d(x,y) = i$, there are precisely c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$, where we define $b_D = c_0 = 0$. In particular, any distance-regular graph is regular with valency $k := b_0$. Note that a (non-complete) connected *strongly regular graph* is just a distance-regular graph with diameter two. We define $a_i := k - b_i - c_i$ for notational convenience. Note

that $a_i =\mid \Gamma(y) \cap \Gamma_i(x) \mid$ holds for any two vertices x, y with $d(x, y) = i$ $(0 \leq i \leq D)$.

For a distance-regular graph Γ and a vertex $x \in V(\Gamma)$, we denote $k_i := |\Gamma_i(x)|$ and $p^i_{jh} := |\{w | w \in \Gamma_j(x) \cap \Gamma_h(y)\}|$ for any $y \in \Gamma_i(x)$. It is easy to see that $k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$ and hence it does not depend on x. The numbers a_i, b_{i-1} and c_i $(1 \leq i \leq D)$ are called the *intersection numbers*, and the array $\{b_0, b_1, \cdots, b_{D-1}; c_1, c_2, \cdots, c_D\}$ is called the *intersection array* of Γ. A distance-regular graph with intersection array $\{k, \mu, 1; 1, \mu, k\}$ is called a *Taylor graph*.

Suppose that Γ is a distance-regular graph with valency $k \geq 2$ and diameter $D \geq 2$, and let A_i be the matrix of Γ such that the rows and the columns of A_i are indexed by the vertices of Γ and the (x, y)-entry is 1 whenever x and y are at distance i and 0 otherwise. We will denote the adjacency matrix of Γ as A instead of A_1. The eigenvalues of the graph Γ are the eigenvalues of A.

The Bose-Mesner algebra M for a distance-regular graph Γ is the matrix algebra generated by the adjacency matrix A of Γ. A basis of M is $\{A_i \mid i = 0, \cdots, D\}$, where $A_0 = I$. The algebra M has also a basis consisting of primitive idempotents $\{E_0 = \frac{1}{v}J, E_1, \cdots, E_D\}$, and E_i is the orthogonal projection onto the eigenspace of θ_i. Note that M is closed under the componentwise multiplication $\circ$. Now, let numbers q^k_{ij} $(0 \leq i, j, k \leq D)$ be defined by $E_i \circ E_j = \frac{1}{v} \sum_{k=0}^{D} q^k_{ij} E_k$. The numbers q^k_{ij} $(0 \leq i, j, k \leq D)$ are called the *Krein parameters* of Γ and are always non-negative by Delsarte (cf. [6, Theorem 2.3.2]).

Some standard properties of the intersection numbers are collected in the following lemma.

Lemma 1 ([6, Proposition 4.1.6])
Let Γ be a distance-regular graph with valency k and diameter D. Then the following holds:
(i) $k = b_0 > b_1 \geq \cdots \geq b_{D-1}$;
(ii) $1 = c_1 \leq c_2 \leq \cdots \leq c_D$;
(iii) $b_i \geq c_j$ if $i + j \leq D$.

Suppose that Γ is a distance-regular graph with valency $k \geq 2$ and diameter $D \geq 1$. Then Γ has exactly $D + 1$ distinct eigenvalues, namely $k = \theta_0 > \theta_1 > \cdots > \theta_D$ ([6, p.128]), and the multiplicity of θ_i $(0 \leq i \leq D)$ is denote by m_i. For an eigenvalue θ of Γ, the sequence $(u_i)_{i=0,1,\ldots,D} = (u_i(\theta))_{i=0,1,\ldots,D}$ satisfying $u_0 = u_0(\theta) = 1$, $u_1 = u_1(\theta) = \theta/k$, and

$$c_i u_{i-1} + a_i u_i + b_i u_{i+1} = \theta u_i \ (i = 2, 3, \ldots, D - 1)$$

is called the *standard sequence* corresponding to the eigenvalue θ ([6, p.128]). A sign change of $(u_i)_{i=0,1,\ldots,D}$ is a pair (i, j) with $0 \leq i < j \leq D$ such that $u_i u_j < 0$ and $u_t = 0$ for $i < t < j$.

In this paper we say that an intersection array is *feasible* if it satisfies the following four conditions:

(i) all its intersection numbers are integral;

(ii) all the multiplicities are positive integers;

(iii) for any $0 \leq i \leq D$, $k_i a_i$ is even;

(iv) all Krein parameters are non-negative.

Recall that a *clique* of a graph is a set of mutually adjacent vertices and that a *co-clique* of a graph is a set of vertices with no edges. A clique $\mathcal{C}$ of a distance-regular graph with valency k, diameter $D \geq 2$ and smallest eigenvalue θ_D, is called *Delsarte clique* if $\mathcal{C}$ contains exactly $1 - \frac{k}{\theta_D}$ vertices. The *strong product* $G \boxtimes H$ of graphs G and H is a graph such that the vertex set of $G \boxtimes H$ is the Cartesian product $V(G) \times V(H)$ and any two different vertices (u, v) and (u', v') are adjacent in $G \boxtimes H$ if and only if ($u = u'$ or u is adjacent to u') and ($v = v'$ or v is adjacent to v'). For a given positive integer s, the *s-clique extension* of a graph G is the strong product $G \boxtimes K_s$ of G and K_s, where K_s is the complete graph (or clique) of size s.

A graph Γ is called *graph of order* (s, t) if Γ is locally disjoint union of $t+1$ copies of $(s+1)$-cliques. Note that, if Γ is a distance-regular graph with $c_2 = 1$ and valency k, then Γ is a graph of order (s, t) for some $s (= a_1)$ and t, and hence the valency k is equal to $(s+1)(t+1)$. A *Terwilliger graph* is a connected non-complete graph Γ such that, for any two vertices u, v at distance two, the subgraph induced on $\Gamma(u) \cap \Gamma(v)$ in Γ is a clique of size μ (for some fixed $\mu \geq 1$). A graph Γ is called *bipartite* if it has no odd cycle. (If Γ is a distance-regular graph with diameter D and bipartite, then $a_1 = a_2 = \ldots = a_D = 0$.)

An *antipodal* graph is a connected graph Γ with diameter $D > 1$ for which being at distance 0 or D is an equivalence relation. If, moreover, all equivalence classes have the same size r, then Γ is also called an *antipodal r-cover*.

Recall the following interlacing result.

Theorem 2 *(cf.[10]) Let $m \leq n$ be two positive integers. Let A be an $n \times n$ matrix, that is similar to a (real) symmetric matrix, and let B be a principal $m \times m$ submatrix of A. Then, for $i = 1, \ldots, m$,*

$$\theta_{n-m+i}(A) \leq \theta_i(B) \leq \theta_i(A)$$

holds, where A has eigenvalues $\theta_1(A) \geq \theta_2(A) \geq \ldots \geq \theta_n(A)$ and B has eigenvalues $\theta_1(B) \geq \theta_2(B) \geq \ldots \geq \theta_m(B)$.

For the convenience of the reader, we give a proof of the following lemma.

Lemma 3 ([8, Lemma 3.1]) Let Γ be a distance-regular graph with diameter D and valency k. If $D \geq 3$, then $b_1 \geq \frac{1}{3}k + \frac{1}{3}$.

Proof: If $b_1 < \frac{1}{3}k + \frac{1}{3}$, then $a_1 + 1 > \frac{2}{3}k - \frac{1}{3}$. Let x be a vertex of Γ. As $\Delta(x)$ is not a complete graph, $\Delta(x)$ has non-adjacent vertices. So, $2(a_1 + 1) - (c_2 - 1) \leq k$, and hence $c_2 \geq 2(a_1 + 1) - k + 1 > \frac{1}{3}k + \frac{1}{3} > b_1$. Lemma 1 (*iii*) implies that $D \leq 2$ and this is a contradiction. □

In [6, Proposition 5.5.1 (ii)], there is an error in the statement when equality occurs. Although this is fixed in [5], for the convenience of the reader, we give a proof for this.

Proposition 4 Let Γ be a distance-regular graph with diameter $D \geq 3$, valency k and intersection number $a_1 > 0$. Then $a_i + a_{i+1} \geq a_1$ for $i = 1, \ldots, D-1$. If $a_i + a_{i+1} = a_1$, then $i = D-1$, $a_D = 0$, $a_{D-1} = a_1$ and $b_{D-1} = 1$.

Proof: Let $i \in \{1, \ldots, D-1\}$ and let x, y be a pair of vertices of Γ at distance $i+1$. Then $|\Gamma_i(x) \cap \Gamma_2(y)| = p^{i+1}_{i,2} = \frac{c_{i+1}(a_i + a_{i+1} - a_1)}{c_2} \geq 0$ which gives $a_i + a_{i+1} \geq a_1$. If $a_i + a_{i+1} = a_1$, then we have $i = D-1$ by [6, Proposition 5.5.1 (i)], and hence $p^D_{D-1,2} = 0$. This implies that each vertex of $\Gamma(y) \cap \Gamma_{D-1}(x)$ is adjacent to each vertex of $\Gamma(y) \cap \Gamma_D(x)$. We will show that $a_D = 0$ by way of contradiction. Assume $a_D > 0$. This implies that the complement of the local graph at y, say $\overline{\Delta(y)}$, is disconnected. Let C be a connected component of $\overline{\Delta(y)}$. If C is a singleton, then $a_1 = k-1$, and hence Γ is complete. So we have $|C| \geq 2$. Let z and w be two adjacent vertices in C. Then they have at most $c_2 - 1$ common neighbors in $\Delta(y)$, as z and w are not adjacent in Γ. This means that C has size at least $k - c_2 + 1$. This means, $k = |\overline{\Delta(y)}| \geq 2(k - c_2 + 1)$, and hence $c_2 \geq \frac{1}{2}k + 1$. On the other hand, $\overline{\Delta(y)}$ is $(k - a_1 - 1)$-regular, and hence C has size at least $k - a_1$ and we obtain $k = |\overline{\Delta(y)}| \geq 2(k - a_1)$. This means that $a_1 \geq \frac{1}{2}k$, and hence $b_1 \leq \frac{1}{2}k - 1 < c_2$. This contradicts $D \geq 3$. Therefore $a_D = 0$. Now we show that $b_{D-1} = 1$. Let z be a vertex of $\Gamma_{D-1}(x)$. If $b_{D-1} > 1$, then for any two vertices u and v of $\Gamma(z) \cap \Gamma_D(x)$, we have $a_2 = |\Gamma(u) \cap \Gamma_2(v)| = 0$ as $a_D = 0$ and $p^D_{D-1,2} = 0$. But, by [6, Proposition 5.5.6], we have $a_2 \geq \min\{b_2, c_2\} \geq 1$, as $a_1 > 0$ and $D \geq 3$. This is a contradiction. So, $b_{D-1} = 1$. □

3 Distance-regular graphs with large a_1

In this section we classify the distance-regular graphs with $a_1 \geq \frac{1}{2}k - 1$ and diameter $D \geq 3$. First we show that if c_2 or b_2 is large, then $D = 3$ and imprimitive, where imprimitive means that the graphs are either bipartite or antipodal. This generalizes [8, Lemma 3.14].

Proposition 5 Let Γ be a distance-regular graph with diameter $D \geq 3$ and valency k. If $c_2 > \frac{1}{2}k$ or $b_2 > \frac{1}{2}k_3$, then $D = 3$ and Γ is either bipartite or a Taylor graph.

Proof: Assume $c_2 > \frac{1}{2}k$ or $b_2 > \frac{1}{2}k_3$. Then $d(y, w) \leq 2$ for a fixed vertex x and $y, w \in \Gamma_2(x)$. Then $p^2_{23} = 0$ which in turn implies $\frac{c_3(a_2+a_3-a_1)}{c_2} = p^3_{22} = 0$ and hence $a_1 = a_2 + a_3$. If $a_1 \neq 0$, then, by Proposition 4, $D = 3$ and Γ is a Taylor graph. So, we may assume $a_1 = 0$. Then $a_2 = a_3 = 0$. Furthermore, if $b_2 > \frac{1}{2}k_3$ and $c_2 \leq \frac{1}{2}k$, then $k_3 = \frac{kb_1b_2}{c_2c_3} > \frac{k(k-1)(k_3/2)}{(k/2)c_3} = \frac{(k-1)k_3}{c_3}$. So, $c_3 > k - 1$ and hence $D = 3$ and Γ is bipartite. This shows that we may assume $c_2 > \frac{1}{2}k$ but this again implies $D = 3$ (and Γ is bipartite) by Lemma 1 (iii) as $c_2 > b_2$. $\square$

Now we show that the distance-regular Terwilliger graphs with large a_1 and $c_2 \geq 2$ are known.

Proposition 6

(i) Let Γ be a connected non-complete strongly regular graph with valency k. If $c_2 = 1$ and $k < 7(a_1 + 1)$, then Γ is either the pentagon or the Petersen graph.

(ii) Let Γ be a connected non-complete strongly regular Terwilliger graph with v vertices and valency k. If $v \leq 7k$ then Γ is either the pentagon or the Petersen graph.

(iii) Let Γ be a distance-regular Terwilliger graph with v vertices, valency k and diameter D. If $k \leq (6 + \frac{8}{57})(a_1 + 1)$ and $c_2 \geq 2$, then Γ is the icosahedron, the Doro graph (see [6, Section 12.1]) or the Conway-Smith graph (see [6, Section 13.2]).

Proof:

(i) Since $c_2 = 1$, then $a_1 + 1$ divides k, and we obtain $(t+1)(a_1+1) = k < 7(a_1+1)$ with $t \in \{1, 2, 3, 4, 5\}$. Now we will show that $a_1 < t$. Let $\mathbf{C}$ be the set of all $(a_1 + 2)$-cliques in the graph Γ. By counting the number of pairs $(x, \mathcal{C})$, where x is a vertex of the graph Γ and $\mathcal{C}$ is a clique of $\mathbf{C}$ containing x, in two ways, we have $|V(\Gamma)|(t + 1) = (a_1 + 2)|\mathbf{C}|$.

Suppose that $a_1 \geq t$, then $|\mathbf{C}| < |V(\Gamma)|$. Let M be the vertex-($(a_1 + 2)$-clique) incidence matrix of Γ, i.e. M is the 01-matrix whose rows and columns are indexed by the vertices and $(a_1 + 2)$-cliques of Γ, respectively, and the $(x, \mathcal{C})$-entry of M is 1 whenever the vertex x is in the clique $\mathcal{C}$ and 0 otherwise. Then MM^T is a singular matrix, as $|\mathbf{C}| < |V(\Gamma)|$, and hence $-t - 1$ is an eigenvalue of Γ, as $MM^T = A + (t+1)I$, where A is the adjacency matrix of Γ. As $-t-1$ is an eigenvalue of Γ, by [6, Proposition 4.4.6], any clique $\mathcal{C}$ of the set $\mathbf{C}$ is a Delsarte clique, and hence for all $x \in V(\Gamma)$ and all $\mathcal{C} \in \mathbf{C}$, there exist $y \in \mathcal{C}$

such that $d(x,y) \le 1$ by [1, Lemma 3]. By considering two vertices u and v at distance two, and the $t+1$ cliques cliques containing u, it follows that u and v have at least $t+1$ common neighbors, which is a contradiction.

So, $0 \le a_1 < t \le 5$ holds. But except for the cases $(t,a_1) = (1,0)$ and $(t,a_1) = (2,0)$, a strongly regular graph does not exist, as the multiplicity of the second largest eigenvalue is not an integer. For $(t,a_1) = (1,0)$ and $(t,a_1) = (2,0)$, we obtain the pentagon and the Petersen graph, respectively.

(ii) As $1 + k + \frac{b_1}{c_2}k < v \le 7k$, it follows that $b_1 < 6c_2$. If $c_2 = 1$, then by *(i)*, Γ is either the pentagon or the Petersen graph. So, we may assume that $c_2 > 1$. Note that $k \ge 2(a_1+1) - (c_2-1)$ (as Γ is not a complete graph), which implies that $13c_2 > c_2 + 2b_1 \ge k+1$, and hence $k < 13c_2 - 1 < 50(c_2-1)$, as $c_2 \ge 2$. So, there are no such strongly regular Terwilliger graph with $c_2 > 1$ by [6, Corollary 1.16.6].

(iii) Let x be a vertex of Γ. Then the local graph $\Delta(x)$ at x is an s-clique extension of a strongly regular Terwilliger graph Σ with parameters $\bar{v} = \frac{k}{s}$, $\bar{k} = \frac{a_1-s+1}{s}$, and $\bar{c_2} = \frac{c_2-1}{s}$ by [6, Theorem 1.16.3]. As $\bar{c_2} \ge 1$ ($c_2 \ge 2$), we have $c_2 - 1 \ge s$. If Σ is the pentagon or the Petersen graph, then $\Delta(x) = \Sigma$ and $s = 1$, and hence by [6, Theorem 1.16.5], we are done in this case. So we may assume that $\bar{v} > 7\bar{k}$ (by *(ii)*). As $k \le (6 + \frac{8}{57})(a_1+1)$, we obtain $s > \frac{7}{57}(a_1+1) \ge \frac{1}{50}k$. Now, as $c_2 - 1 \ge s$, it follows $k < 50(c_2-1)$, and hence we are done by [6, Corollary 1.16.6].

□

Remark 1. There exist generalized quadrangles of order (q,q) for any prime power q (see, for example [9, p.83]). Note that the flag graph of any generalized quadrangle of order (q,q) is a distance-regular graph with $k = 2q$ and $c_2 = 1$. This shows that there are infinitely many distance-regular graphs with $a_1 > \frac{1}{7}k$. See also, Theorem 11 below.

Before we classify the distance-regular graphs with $a_1 \ge \frac{1}{2}k - 1$, we first introduce some results for this classification.

Lemma 7 Let Γ be a distance-regular graph with diameter $D \ge 3$ and valency k. If $a_1 \ge \frac{1}{2}k - 1$ and $c_2 \ge 2$, then $b_2 < c_2$, and hence $D = 3$.

Proof: If Γ is a Terwilliger graph, then Γ is the icosahedron by Proposition 6 *(iii)*. If Γ is not a Terwilliger graph, then Γ has a quadrangle. Then by [6, Theorem 5.2.1], $c_2 - b_2 \ge c_1 - b_1 + a_1 + 2 \ge 2$ holds, and hence $D = 3$ by Lemma 1 *(iii)*. □

3.1 Some eigenvalues results

In the next three lemmas, we give some results on the eigenvalues of a distance-regular graph.

Lemma 8 Let Γ be a distance-regular graph with diameter three and distinct eigenvalues $k = \theta_0 > \theta_1 > \theta_2 > \theta_3$. If $a_3 = 0$, then $\theta_1 > 0 > -1 \geq \theta_2 \geq -b_2 \geq \theta_3$.

Proof: As $a_3 = 0$, we know that θ_1, θ_2 and θ_3 are the eigenvalues of

$$T := \begin{bmatrix} -1 & b_1 & 0 \\ 1 & k - b_1 - c_2 & b_2 \\ 0 & c_2 & -b_2 \end{bmatrix}$$

by [6, p.130]. Since the principal submatrix $\begin{bmatrix} -1 & 0 \\ 0 & -b_2 \end{bmatrix}$ of T has eigenvalues -1 and $-b_2$, it follows that the inequality $\theta_1 \geq -1 \geq \theta_2 \geq -b_2 \geq \theta_3$ holds by Theorem 2. As Γ has an induced path P of length three, $\theta_1 \geq$ second largest eigenvalue of P, which is greater than zero.

□

Lemma 9 Let Γ be a distance-regular graph with diameter $D \geq 3$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. If Γ has an eigenvalue θ with multiplicity smaller than $\frac{1}{2}k$ then the following holds:
(1) $\theta \in \{\theta_1, \theta_D\}$,
(2) θ is integral,
(3) $\theta + 1$ divides b_1.

Proof: This lemma follows immediately from [6, Theorem 4.4.4]. □

Lemma 10 Let Γ be a distance-regular graph with diameter $D \geq 3$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. If Γ has an eigenvalue θ with multiplicity at most $k - 2$ then the following holds:
(1) $\theta_2 \geq -1 - \frac{b_1}{\theta+1}$ if $\theta = \theta_1$,
(2) $\theta_{D-1} \leq -1 - \frac{b_1}{\theta+1}$ if $\theta = \theta_D$.

Proof: Let x be a vertex of Γ. Then the local graph $\Delta(x)$ has smallest eigenvalue at least $-1 - \frac{b_1}{\theta_1+1}$ and second largest eigenvalue at most $-1 - \frac{b_1}{\theta_D+1}$, by [6, Theorem 4.4.3]. As $\Delta(x)$ has k eigenvalues at least $-1 - \frac{b_1}{\theta_1+1}$ and $k-1$ eigenvalues at most $-1 - \frac{b_1}{\theta_D+1}$, by Theorem 2, the inequalities follow. □

Lemma 10 gives evidence for the following conjecture.

Conjecture A. Let Γ be a distance-regular graph with diameter three and distinct eigenvalues $k = \theta_0 > \theta_1 > \theta_2 > \theta_3$. Then

$$-1 - \frac{b_1}{\theta_3 + 1} \geq \theta_2 \geq -1 - \frac{b_1}{\theta_1 + 1}.$$

Remark 2. Conjecture A is true when Γ is bipartite, as then $b_1 = k - 1$, $\theta_1 = -\theta_2 = \sqrt{b_2}$ and $\theta_3 = -k$. Hence $0 = -1 - \frac{b_1}{\theta_3+1} > \theta_2 = -\sqrt{b_2} > -1 - \frac{b_1}{\theta_1+1}$, where the last inequality holds, as $k > b_2$. Conjecture A is also true when Γ is antipodal, as then $\theta_2 = -1$. We also checked that all the feasible intersection arrays in the table of primitive distance-regular graphs with diameter three [6, p. 425-431], satisfy this conjecture.

Remark 3. For a distance-regular graph with diameter $D \geq 4$, we have $\theta_2 \geq 0$ by Theorem 2.

3.2 Classification of distance-regular graphs with $\mathbf{a_1 \geq \frac{1}{2}k - 1}$

Now we are ready to classify the distance-regular graphs with $a_1 \geq \frac{1}{2}k - 1$.

Theorem 11 Let Γ be a distance-regular graph with diameter $D \geq 3$ and valency k. If $a_1 \geq \frac{1}{2}k - 1$, then one of the following holds:
(1) Γ is a polygon,
(2) Γ is the line graph of a Moore graph,
(3) Γ is the flag graph of a regular generalized D-gon of order (s, s) for some s,
(4) Γ is a Taylor graph,
(5) Γ is the Johnson graph $J(7, 3)$,
(6) Γ is the halved 7-cube.

Proof: If Γ does not contain a $K_{1,3}$ as an induced subgraph of Γ, then one of (1), (2), (3) or (4) holds by [4, Theorem 1.2]. So, we may assume that Γ contains a $K_{1,3}$. Since Γ contains a $K_{1,3}$, we find $3(a_1 + 1) - 3(c_2 - 1) \leq k$, and hence $c_2 \geq \frac{1}{6}k + 1$. So, $c_2 \geq 2$ and it follows that $D = 3$ by Lemma 7. We will consider two cases, namely the case $a_3 = 0$ and the case $a_3 \neq 0$.

Case 1) First let us assume $a_3 = 0$. Then $c_3 = k$ and it follows $k_3 = \frac{kb_1b_2}{c_2c_3} = \frac{b_1b_2}{c_2}$. Let x be a vertex of Γ and y be a vertex of $\Gamma_3(x)$. Then $\Gamma_3(x)$ contains $\frac{k(b_2-1)}{c_2}$ vertices which are at distance 2 from y, as $a_3 = 0$. Hence $b_1b_2 > k(b_2 - 1)$ and this implies $b_2 = 1$, as $b_1 \leq \frac{1}{2}k$. Thus, $k_3 \in \{1, 2\}$, as $b_1 \leq \frac{1}{2}k$ and $c_2 > \frac{1}{6}k$.

(1) If $k_3 = 1$, then Γ is a Taylor graph.

(2) If $k_3 = 2$, then $b_1 = 2c_2$, and hence $v = 3k+3$, as $k_2 = 2k$. Since $b_1 \leq \frac{1}{2}k$, we have $c_2 \leq \frac{1}{4}k$, which implies $a_2 \geq \frac{3}{4}k - 1$, as $b_2 = 1$. Let $k > \theta_1 > \theta_2 > \theta_3$ be the distinct eigenvalues of Γ. Then $k + \theta_1 + \theta_2 + \theta_3 = a_1 + a_2 + a_3 \geq (\frac{1}{2}k - 1) + (\frac{3}{4}k - 1)$ implies $\theta_1 + \theta_2 + \theta_3 \geq \frac{1}{4}k - 2$. As $a_3 = 0$ and $b_2 = 1$, we have $\theta_2 = -1 > \theta_3$ by Lemma 8, and hence $\theta_1 > \frac{1}{4}k$.

If the multiplicity m_1 of θ_1 is smaller than $\frac{1}{2}k$, then by Lemma 9, $\frac{b_1}{\theta_1+1}$ is an integer and hence $\theta_1 = b_1 - 1$, as $b_1 \leq \frac{1}{2}k$ and $\theta_1 > \frac{1}{4}k$. Then there is no such graph by [6, Theorem 4.4.11].

If $m_1 \geq \frac{1}{2}k$, then $4k^2 \geq (3k+3)k = vk = k^2 + m_1\theta_1^2 + m_2\theta_2^2 + m_3\theta_3^2 > k^2 + \frac{1}{2}k(\frac{1}{4}k)^2$, and hence $k < 96$.

We checked by computer the feasible intersection arrays of antipodal 3-covers with diameter three, satisfying $k < 96$, $a_1 \geq \frac{1}{2}k - 1$ and $c_2 > \frac{1}{6}k$, and no intersection arrays were feasible .

Case 2) Now we assume $a_3 \neq 0$. We first will show that k in this case $k \leq 945$ holds. If $\theta_1 \geq \frac{k}{4}$ holds, then similarly as in (2) of **Case 1)**, we can show that $k < 96$ or $\theta_1 = b_1 - 1$. If $\theta_1 = b_1 - 1 \geq \frac{k}{4}$ holds, then Γ is either the Johnson graph $J(7,3)$ or the halved 7-cube by [6, Theorem 4.4.11]. So, we find that if $\theta_1 \geq \frac{k}{4}$ holds, then we have $k < 96$. So we may assume that $\theta_1 < \frac{k}{4}$. As $\theta_1 \geq \min\{\frac{a_1+\sqrt{a_1^2+4k}}{2}, a_3\}$ ([11, Lemma 6]) and $\frac{a_1+\sqrt{a_1^2+4k}}{2} > \frac{k}{4}$, we have $a_3 < \frac{k}{4}$, and hence $c_3 > \frac{3}{4}k$. As $a_3 \neq 0$, Γ is not a Taylor graph. Then by Proposition 5, we find $\frac{\frac{k}{2}}{c_2}k \geq \frac{b_1}{c_2}k = k_2 \geq 2c_3 > \frac{3}{2}k$, which implies $c_2 < \frac{k}{3}$. Let η_1=$\max\{-1, a_3 - b_2\}$ and η_2=$\min\{-1, a_3 - b_2\}$. Then $\theta_1 \geq \eta_1 \geq \theta_2 \geq \eta_2 \geq \theta_3$ by [12, Proposition 3.2]. Now we will show that $\theta_1 \geq \frac{k}{2} - c_2$.
If $\theta_2 \geq 0$, then $\eta_1 = a_3 - b_2$ and $\eta_2 = -1$, and hence $a_3 \geq b_2 + \theta_2$ and $\theta_3 \leq -1$. We find $\theta_1 + \theta_2 + \theta_3 = a_1 + a_2 + a_3 - k \geq \frac{-k}{2} - 1 + a_2 + b_2 + \theta_2$ and this implies $\theta_1 \geq a_2 + b_2 - \frac{k}{2} = \frac{k}{2} - c_2$.
If $\theta_2 < 0$, then $\theta_3 \leq a_3 - b_2$ implies $a_3 \geq b_2 + \theta_3$. We find $\theta_1 + \theta_2 + \theta_3 = a_1 + a_2 + a_3 - k \geq \frac{-k}{2} - 1 + a_2 + b_2 + \theta_3$ and this implies $\theta_1 \geq a_2 + b_2 - \frac{k}{2} = \frac{k}{2} - c_2$ as $\theta_2 \leq -1$ or $a_3 - b_2 \geq \theta_2 \geq -1 \geq \theta_3$. So, we have shown $\theta_1 \geq \frac{k}{2} - c_2$ and this implies $\theta_1 > \frac{k}{6}$ and $c_2 > \frac{k}{4}$, as $c_2 < \frac{k}{3}$ and $\theta_1 < \frac{k}{4}$ respectively.

If the multiplicity m_1 of θ_1 is smaller than $\frac{1}{2}k$, then by Lemma 9, $\frac{b_1}{\theta_1+1}$ is an integer and hence $\frac{b_1}{\theta_1+1} \in \{1, 2\}$, as $b_1 \leq \frac{1}{2}k$ and $\theta_1 > \frac{1}{6}k$.

If $\frac{b_1}{\theta_1+1} = 1$, then there is no such graph by [6, Theorem 4.4.11].

If $\frac{b_1}{\theta_1+1} = 2$, then $u_2(\theta_1) = \frac{(\theta_1 - a_1)u_1(\theta_1) - 1}{b_1} = \frac{\frac{3b_1}{4} - \frac{k}{2} - \frac{3}{2}}{k} < -\frac{1}{8}$, as $\theta_1 - a_1 = \frac{b_1}{2} - 1 - a_1 = \frac{3}{2}b_1 - k$ and $u_1(\theta_1) = \frac{\theta_1}{k} = \frac{\frac{b_1}{2} - 1}{k}$. As $\theta_1 = a_3$ implies $u_2(\theta_1) = 0$, we assume $\theta_1 \neq a_3$. Then $c_3u_2(\theta_1) + a_3u_3(\theta_1) = \theta_1 u_3(\theta_1)$ follows that $u_3(\theta_1) = \frac{c_3}{\theta_1 - a_3}u_2(\theta_1)$. As $u_3(\theta_1) \geq -1$, we find $\theta_1 - a_3 > \frac{3}{32}k$ and hence $a_3 < \frac{5}{32}k$. As

$\frac{b_1}{2}-1=\theta_1\geq\frac{1}{2}k-c_2$ and $k_2\geq 2c_3$, we find that $\theta_1\leq\frac{9}{40}k$ implies $c_3\leq\frac{37}{44}k$, unless $k\leq 160$. This shows that $k\leq 160$ or $\theta_1>\frac{9}{40}k$. So, we may assume $\theta_1>\frac{9}{40}k$ and $a_3<\frac{5}{32}k$. As $a_1\geq\frac{1}{2}k-1$ and $c_2\geq 2$, we have $b_2<c_2$ and this shows $k_3=k\frac{b_1b_2}{c_2c_3}<k\frac{b_1}{c_3}<k\frac{k/2}{27k/32}=\frac{16}{27}k$. Then by [6, Theorem 4.1.4], we find $m_1=\frac{v}{\sum_{i=0}^3 u_i(\theta_1)^2k_i}<\frac{k+k_2+16k/27}{(9/40)^2k+(-1/8)^k_2}=\frac{43k/27+k_2}{81k/1600+k_2/64}$. As $\frac{k}{3}\leq b_1\leq\frac{k}{2}$ and $\frac{k}{4}<c_2<\frac{k}{3}$, we have $k\leq k_2<2k$ and $\frac{43k/27+k_2}{81k/1600+k_2/64}$ has maximum value smaller than 43.88 when $k_2=2k$. So, $m_1\leq 43$. Thus, we have $k\leq\frac{(43+2)(43-1)}{2}=945$ by [6, Theorem 5.3.2].

If $m_1\geq\frac{1}{2}k$, then similarly as in (2) of **Case 1)** we obtain $k<312$.

We checked by computer the feasible intersection arrays of distance-regular graphs with diameter three satisfying $k\leq 945$, $a_1\geq\frac{1}{2}k-1$, $c_2>\frac{1}{6}k$ and $a_3\neq 0$, and only the intersection arrays of the Johnson graph $J(7,3)$ and the halved 7-cube, were feasible.

□

4 Distance-regular graphs with small $\mathbf{k_2}$

In this section we give two results on the distance-regular graphs with small k_2. In the first result we look at $k_2<2k$, and in the second result we classify the distance-regular graphs with $k_2\leq\frac{3}{2}k$ and diameter at least three.

Theorem 12 Let $\varepsilon>0$. Then there exist a real number $\kappa=\kappa(\varepsilon)\geq 3$ such that if Γ is a distance-regular graph with diameter $D\geq 3$, valency $k\geq\kappa(\varepsilon)$ and $k_2\leq(2-\varepsilon)k$, then $D=3$ and Γ is either bipartite or a Taylor graph.

Proof: We may assume $k\geq 3$. If $c_2=1$, then $b_1=1$, as $k\leq k_2=\frac{kb_1}{c_2}\leq(2-\varepsilon)k$ and this implies $a_1=k-2$. As $c_2=1$, we know that $a_1+1=k-1$ divides k, which gives $k=2$, a contradiction to $k\geq 3$. So, we may assume $c_2\geq 2$. Suppose Γ either has $D\geq 4$ or ($D=3$ and Γ is not bipartite or a Taylor graph), then by Proposition 5, $c_2\leq\frac{1}{2}k$ and $b_2\leq\frac{1}{2}k_3$. As $c_2\leq\frac{1}{2}k$ (respectively $b_2\leq\frac{1}{2}k_3$), we have $(2-\varepsilon)k\geq k_2\geq 2b_1$ (respectively $(2-\varepsilon)k\geq k_2\geq 2c_3$), and hence $a_1\geq\frac{\varepsilon}{2}k-1$ (respectively $a_3\geq\frac{\varepsilon}{2}k$). So, by [11, Lemma 6], we have $\theta_1\geq\min\{\frac{a_1+\sqrt{a_1^2+4k}}{2},a_3\}\geq\min\{a_1+1,a_3\}\geq\frac{\varepsilon}{2}k$. This implies $u_1(\theta_1)=\frac{\theta_1}{k}\geq\frac{\varepsilon}{2}$. As $2k>(2-\varepsilon)k\geq k_2=\frac{kb_1}{c_2}$, we find $2c_2>b_1$. Then, by [2, Lemma 5.2], we obtain $D\leq 4$. For $D=3$, it is easy to check $v\leq 7k$. If $D=4$, then by [6, Theorem 5.4.1], $c_3\geq\frac{3}{2}c_2$ and this implies $k_3=k_2\frac{b_2}{c_3}\leq k_2\frac{b_1}{3/2c_2}<\frac{8}{3}k$ and $k_4<\frac{32}{9}k$, as $\frac{b_1}{c_2}<2$. So, $v\leq 10k$. Then, by [6, Theorem 4.1.4], the multiplicity m_1 of θ_1 is smaller than $\frac{v}{u_1(\theta_1)^2k}<\frac{40}{\varepsilon^2}$. So, $k<\frac{(40/\varepsilon^2-1)(40/\varepsilon^2+2)}{2}$ (by [6, Theorem 5.3.2]).

Thus, if we take $\kappa(\varepsilon) = \frac{(40/\varepsilon^2-1)(40/\varepsilon^2+2)}{2}$, then $D = 3$ and Γ is either bipartite or a Taylor graph. $\square$

Remark 4. The Hadamard graphs have intersection array $\{k, k-1, \frac{1}{2}k, 1; 1, \frac{1}{2}k, k-1, k\}$ and have $k_2 = 2(k-1)$. For $k = 2^t$ $(t = 1, 2, \ldots)$, there exists a Hadamard graph (see, for example [6, Section 1.8]). This shows that the above theorem is quite sharp.

Question. For fixed positive constant C, are there only finitely many primitive distance-regular graphs with diameter at least three, valency $k \geq 3$ and $k_2 < Ck$?

In the next theorem, we classify the distance-regular graphs with $k_2 \leq \frac{3}{2}k$ and diameter at least three.

Theorem 13 Let Γ be a distance-regular graph with diameter $D \geq 3$, v vertices and valency k. If $k_2 \leq \frac{3}{2}k$, then one of the following holds:
(1) $k = 2$ and Γ is a polygon,
(2) $D = 3$ and Γ is bipartite,
(3) $D = 3$ and Γ is a Taylor graph,
(4) Γ is the Johnson graph $J(7, 3)$,
(5) Γ is the 4-cube.

Proof: If $c_2 = 1$, then $b_1 = 1$, as $k \leq k_2 = \frac{kb_1}{c_2} \leq \frac{3}{2}k$ and this implies $a_1 = k - 2$. As $c_2 = 1$, it follows that $a_1 + 1 = k - 1$ divides k, which gives $k = 2$ and Γ is a polygon. So, we may assume $\frac{1}{2}k \geq c_2 \geq 2$, as $c_2 > \frac{1}{2}k$ implies that $D = 3$ and Γ is either bipartite or a Taylor graph (by Proposition 5). As the distance-regular line graphs with $c_2 \geq 2$ have $D = 2$ (see [6, Theorem 4.2.16]), $c_2 \geq 2$ implies that when $a_1 \geq \frac{1}{2}k - 1$ the graph Γ is either a Taylor graph, the Johnson graph $J(7, 3)$ or the halved 7-cube by Theorem 11, but the halved 7-cube has $k = 12$ and $k_2 = 35$.

Hence, we may assume $\frac{1}{2}k \geq c_2 \geq 2$ and $a_1 < \frac{1}{2}k - 1$. This implies $b_1 > \frac{1}{2}k$, and hence $c_2 \geq \frac{2}{3}b_1 > \frac{1}{3}k$, as $k_2 \leq \frac{3}{2}k$. If $a_1 = 0$, then $\frac{1}{2}k \geq c_2 \geq \frac{2}{3}(k-1)$, which implies $k \leq 4$ and it is easy to check that the theorem holds in this case by [3, 7]. So, we may assume $a_1 > 0$ and this implies $k \geq 5$, as $a_1 < \frac{1}{2}k - 1$. As $a_1 > 0$, we find $a_2 \geq \min\{b_2, c_2\}$, by [6, Proposition 5.5.6], which in turn implies $b_2 < \frac{1}{3}k < c_2$, and hence $D = 3$.

So, from now on, we assume that the diameter D is three, a_1 is positive and the valency k is at least five. Here note that $v \leq \frac{7}{2}k$, as $k_3 \leq k_2\frac{b_2}{c_2} < \frac{3}{2}k\frac{k/6}{k/3} = \frac{3}{4}k$ when $b_2 \leq \frac{1}{6}k$, and $b_2 > \frac{1}{6}k$ implies $c_3 > \frac{1}{2}k$ by [6, Theorem 5.4.1], and hence $k_3 \leq \frac{3}{2}k\frac{k/3}{k/2} = k$. As $k_2 \geq 2b_1$ and $k_2 \geq 2c_3$ (Proposition 5), we find $a_1 \geq \frac{1}{4}k - 1$ and $a_3 \geq \frac{1}{4}k$ respectively. By [11, Lemma 6], we find $\theta_1 \geq \min\{\frac{\mathrm{a_1+\sqrt{a_1^2+4k}}}{2}, \mathrm{a_3}\} \geq \min\{a_1 + 1, a_3\} \geq \frac{1}{4}k$.

If $m_1 \geq \frac{1}{2}k$, then $\frac{7}{2}k^2 \geq vk = k^2 + m_1{\theta_1}^2 + m_2{\theta_2}^2 + m_3{\theta_3}^2 \geq k^2 + \frac{1}{2}k(\frac{1}{4}k)^2$ implies $k \leq 80$. We checked by computer the feasible intersection arrays of distance-regular

graphs with diameter three satisfying $k \leq 80$ and $k_2 \leq \frac{3}{2}k$, and no intersection arrays were feasible.

If $m_1 < \frac{1}{2}k$, then by Lemma 9, $\frac{b_1}{\theta_1+1} \in \{1, 2\}$, as $\theta_1 \geq \frac{1}{4}k$ and $b_1 \leq \frac{3}{4}k$. If $\frac{b_1}{\theta_1+1} = 1$, then there is no such distance-regular graph by [6, Theorem 4.4.11].

For $\frac{b_1}{\theta_1+1} = 2$, we first show that one of $m_1 < 48$ and $k \leq 510$ holds. If $m_1 \geq 48$ and $k > 510$, then by [6, Theorem 4.1.4], $48 \leq m_1 = \frac{v}{\sum_{i=0}^{3} u_i(\theta_1)^2 k_i} < \frac{7k/2}{(\theta_1/k)^2 k}$, as $u_1(\theta_1) = \frac{\theta_1}{k}$ and $v \leq \frac{7}{2}k$. This implies $(\frac{\theta_1}{k})^2 < \frac{7}{96}$ and hence $\theta_1 < (0.271)k$. As $\frac{b_1}{2} - 1 = \theta_1 < (0.271)k$, we have $b_1 < (0.542)k + 2$, which implies $a_1 > (0.458)k - 3$, and hence $a_1 + 1 > (0.458)k - 3 > (0.271)k > \theta_1$, as $k > 510$. This in turn implies $\theta_1 \geq a_3$, as $\theta_1 \geq \min\{a_1 + 1, a_3\}$. Note that if $k_3 < \frac{1}{2}k$, then $v \leq 3k$, as $k_2 \leq \frac{3}{2}k$. As $\theta_1 \geq \frac{1}{4}k$, we find $m_1 = \frac{v}{\sum_{i=0}^{3} u_i(\theta_1)^2 k_i} < \frac{3k}{k/16} = 48$, and this contradicts $m_1 \geq 48$. So, $k_3 \geq \frac{1}{2}k$ and this implies $b_2 \geq \frac{1}{3}c_3$, as $k_3 = k_2\frac{b_2}{c_3}$ and $k_2 \leq \frac{3}{2}k$. Then, as $a_3 \leq \theta_1 < (0.271)k$, we find $c_3 > (0.729)k$, and hence $b_2 > (0.243)k$. Since $a_2 = k - b_2 - c_2$ and $c_2 > \frac{1}{3}k > (0.333)k$, we find $a_2 < (0.424)k$ and this implies $(0.924)k - 1 > a_1 + a_2 \geq k + \theta_2 + \theta_3 \geq k - 3 + \theta_3$, as $\theta_1 \geq a_3$, and by Lemma 10, we have $\theta_2 \geq -3$. So, we obtain $-(0.076)k + 2 \geq \theta_3$. As $k > 510$, we have $-(0.07)k > -(0.076)k + 2 \geq \theta_3$. Here note $m_1 + m_3 \geq k$ by [6, Theorem 4.4.4]. Now $\frac{7}{2}k \geq vk \geq k^2 + m_1{\theta_1}^2 + m_3{\theta_3}^2$, $\theta_1 \geq \frac{1}{4}k$ and $m_1 + m_3 \geq k$ imply $k \leq 510$. This is a contradiction. So, we find that either $m_1 < 48$ or $k \leq 510$. If $m_1 < 48$, then [6, Theorem 5.3.2] implies $k \leq 1127$. In conclusion, we find $k \leq 1127$.

We checked by computer the feasible intersection arrays of the distance-regular graphs with diameter three satisfying $k_2 \leq \frac{3}{2}k$, $\theta_1 = \frac{b_1}{2} - 1$, $m_1 < \frac{1}{2}k$ and $k \leq 1127$, and no intersection arrays were feasible. $\square$

Acknowledgments

We thank E.R. van Dam and S. Bang for their careful reading and comments on an early version of this paper.

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